Numerical semigroups attained by double covers of plane curves of degree six (Algebraic system, Logic, Language and Computer Science)

Komeda, Jiryo

数理解析研究所講究録
第2008巻 2016年 100-106

Numerical semigroups attained by double covers of plane curves of degree six [1]

神奈川工科大学・基礎・教養教育センター　米田 二良
Jiryo Komeda
Center for Basic Education and Integrated Learning
Kanagawa Institute of Technology

Abstract

We investigate Weierstrass semigroups of ramification points on double covers of plane curves of degree 6. We determine all the Weierstrass semigroups, especially when the genus of the covering curve is greater than 29 and the ramification point is on a total flex.

1 Notations and terminologies

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, denoted by $g(H)$. In this paper H always stands for a numerical semigroup. A *curve* means a complete non-singular irreducible algebraic curve over an algebraically closed field k of characteritic 0. For a pointed curve (C, P) we set

$$H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = nP\},$$

where $k(C)$ is the field of rational functions on C. Then $H(P)$ is a numerical semigroup, Here $g(H(P))$ is equal to the genus $g(C)$ of the curve C.

Let C be a plane curve of degree 6 and P_1 its total inflection point, i.e., $T_{P_1}.C = 6P_1$, where T_{P_1} is the tangent line at P_1 on C. Then we have $H(P_1) = \langle 5, 6\rangle$, where for positive integers $a_1, \ldots, a_s$ we denote by $\langle a_1, \ldots, a_s\rangle$ the monoid generated by $a_1, \ldots, a_s$.

If P_2 is a point of C with $T_{P_2}.C = 5P_2 + Q$, $Q \neq P_2$. Then we have $H(P_2) = \langle 5, 9, 13, 17, 21\rangle$. We set $d_2(H) = \{h' \in \mathbb{N}_0 \mid 2h' \in H\}$, which is a numerical semigroup. Let $\pi : \tilde{C} \longrightarrow C$ be a double covering of curves with a ramification point $\tilde{P}$. Then we obtain $d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$. For example, if $C = \mathbb{P}^1$, then we have $H(\tilde{P}) = \langle 2, 2g+1\rangle$ with $g = g(\tilde{C})$ and $d_2(H(\tilde{P})) = \mathbb{N}_0$. We pose the following problem:

DCPHurwitz' Problem. Let C be a plane curve of degree d and $\pi : \tilde{C} \longrightarrow C$ a double covering with a ramification point $\tilde{P}$. Then determine $H(\tilde{P})$.

---

[1] This paper is an extended abstract and the details will appear elsewhere.
This work is a collaboration with Seon Jeong Kim.
The author is partially supported by Grant-in-Aid for Scientific Research (15K04830), Japan Society for the Promotion Science.

Fact. i) In the case of $d = 1, 2$ the result is classical (Hyperelliptic case).
ii) In the case $d = 3$ the result is also classical (Bielliptic case).
iii) In the case $d = 4$ DCPHurwitz' Problem is solved ([6], [1], [2], [3])
iv) Let $d = 5$. If $g(\tilde{C}) \geqq 15$ and $\mathrm{ord}_{\pi(\tilde{P})}(T_{\pi(\tilde{P})}.C) = 5$ or 4, then DCPHurwitz' Problem is solved ([4]) where for a line L in $\mathbb{P}^2$ and $P \in C$ we denote by $\mathrm{ord}_P(L.C)$ the multiplicity of the intersection of L and C at P.

We consider DCPHurwitz' Problem in the case $d = 6$. The following are our results:

Theorem. *Let C be a plane curve of degree* 6 *and $\pi : \tilde{C} \longrightarrow C$ a double covering with a ramification point $\tilde{P}$. Assume that $g(\tilde{C}) \geqq 30$.*
i) *If $\mathrm{ord}_{\pi(\tilde{P})}(T_{\pi(\tilde{P})}.C) = 6$, we determine $H(\tilde{P})$. If we explain it in detail, any numerical semigroup H of genus $g > 29$ with $d_2(H) = \langle 5, 6\rangle$ is given by the Weierstrass semigroup of a ramification point on a double cover of a plane curve of degree* 6.
ii) *If $\mathrm{ord}_{\pi(\tilde{P})}(T_{\pi(\tilde{P})}.C) = 5$, there are eighty two candidates of the numerical semigroups $H(\tilde{P})$. We determine whether seventy four candidates are attained by ramification points on double covers of plane curves of degree* 6 *or not. We do not know whether the remaining eight candidates are attained or not.*

2 The candidates of Weierstrass semigroups of double covers of plane curves of degree 6 over total inflection points

Let H be a numerical semigroup with $d_2(H) = \langle 5, 6\rangle$. We consider the following diagram:

$$\begin{array}{cccccc}
(\rightarrow +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \times & \times & \downarrow \\
(n) & \bullet & \times & \times & \times & +10 \\
 & (n+12) & \bullet & \times & \times & \\
\searrow +12 & & (n+24) & \bullet & \times & \\
 & & & (n+36) & \bullet & \\
 & & & & (n+48) &
\end{array}$$

which is called the triangle Δ associated to the semigroup $2\langle 5, 6\rangle + n\mathbb{N}_0$. The symbol "$\times$" means that it is a non-element of the numerical semigroup larger than n if n is odd $\geqq 21$. The triangle Δ consists of the odd integers $n + 38 - 10i - 2j$ with $0 \leqq i \leqq j \leqq 3$, which do not belong to the numerical semigroup $2\langle 5, 6\rangle + n\mathbb{N}_0$. We associates the coordinate (i, j) to the odd number $n + 38 - 10i - 2j$. The coordinate (k, l) generates the parallelogram

$$P(k, l) = \{(i, j) \mid 0 \leqq i \leqq k, 0 \leqq j \leqq l\}.$$

For example, $P(3, 2)$ is the following, where $\odot$ denotes $n + 38 - 10 * 3 - 2 * 2$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \odot & \times & \times & \downarrow \\
(n) & \bullet & \circ & \circ & \times & 10 \\
 & (n+12) & \bullet & \circ & \circ & \\
\searrow +12 & & (n+24) & \bullet & \circ & \\
 & & & (n+36) & \bullet & \\
 & & & & (n+48) &
\end{array}$$

We can associate to the numerical semigroup H the polygon

$$P(H) = \{(i,j) \in \Delta \mid n + 38 - 10i - 2j \in H\}$$

Let $t(H)$ be the minimum number of generators for H which are odd and distinct from n. Then we have $0 \leqq t(H) \leqq 4$. If $t(H) = 0$, then $P(H) = \emptyset$. If $t(H) = 4$, then $P(H) = \Delta$. If $t(H) = 1$, there are 10 kinds of $P(H)$, i.e., H. For example, if $H = 2\langle 5,6\rangle + \langle n, n+4\rangle$. Then we have $P(H) = P(2,3)$. Ler $r(H)$ be the number of the odd elements of H which are larger than n and less than $n+40$. Then $0 \leqq r(H) \leqq 10$. If $r(H) = 0$, then $P(H) = \emptyset$. If $r(H) = 10$, then $P(H) = \Delta$. If $r(H) = 9$, there are 4 kinds of $P(H)$, i.e., H. For example, if $H = 2\langle 5,6\rangle + \langle n, n+2, n+4, n+6\rangle$, then $P(H)$ is the following:

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \odot & \odot & \odot & \times & \downarrow \\
(n) & \bullet & \circ & \circ & \circ & +10 \\
 & (n+12) & \bullet & \circ & \circ & \\
\searrow +12 & & (n+24) & \bullet & \circ & \\
 & & & (n+36) & \bullet & \\
 & & & & (n+48) &
\end{array}$$

The following table shows the number of H 's with $d_2(H) = \langle 5,6\rangle$ having fixed $t(H)$ and $r(H)$.

$t(H)\backslash r(H)$	0	1	2	3	4	5	6	7	8	9	10	*Total*
0	1	0	0	0	0	0	0	0	0	0	0	1
1	0	1	2	2	3	0	2	0	0	0	0	10
2	0	0	0	1	2	5	4	5	3	0	0	20
3	0	0	0	0	0	0	1	2	2	4	0	9
4	0	0	0	0	0	0	0	0	0	0	1	1
Total	1	1	2	3	5	5	7	7	5	4	1	41

3 The candidates of Weierstrass semigroups of double covers of plane curves of degree 6 over almost total inflection points

Let H be a numerical semigroup with $d_2(H) = \langle 5, 9, 13, 17, 21\rangle$. We consider the following diagram:

$$\begin{array}{cccccc}
(\rightarrow +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \times & \times & \downarrow \\
(n) & \times & \times & \times & \bullet & +10 \\
 & \times & \times & \bullet & (n+18) & \\
 & \times & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

which is called the triangle Δ_a associated to the semigroup $2\langle 5,9,13,17,21\rangle + n\mathbb{N}_0$. The symbol "$\times$" is a non-element of the numerical semigroup larger than n if n is odd $\geqq 21$. The triangle Δ_a consists of the odd integers $n+32-10i+2j$ with $0 \leqq j \leqq i \leqq 3$, which do not belong to the numerical semigroup $2\langle 5,9,13,17,21\rangle + n\mathbb{N}_0$. We associates the coordinate (i,j) to the odd number $n+32-10i+2j$. The coordinate (k,l) generates the parallelogram

$$P_a(k,l) = \{(i,j) \mid 0 \leqq i \leqq k-2(l-j), 0 \leqq j \leqq l\}.$$

For example, $P_a(3,2)$ is the following, where $\odot$ denotes $n+32-10*3+2*2$.

$$\begin{array}{cccccc}
(\rightarrow +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \odot & \times & \downarrow \\
(n) & \times & \times & \circ & \bullet & +10 \\
 & \times & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

We can associate to the numerical semigroup H the polygon

$$P_a(H) = \{(i,j) \in \Delta \mid n+32-10i+2j \in H\}.$$

Let $t(H)$ be the minimum number of generators for H which are odd and distinct from n. Let $r(H)$ be the number of the odd elements of H which are larger than n and less than $n+34$. For example, if $H = 2\langle 5,9,13,17,21\rangle + \langle n, n+4, n+8, n+16\rangle$, then $P_a(H)$ is the following:

$$\begin{array}{cccccc}
(\rightarrow +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \odot & \times & \odot & \downarrow \\
(n) & \times & \circ & \odot & \bullet & +10 \\
 & \circ & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Hence, we have $t(H) = 3$ and $r(H) = 7$.

The following table shows the number of H 's with $d_2(H) = \langle 5,9,13,17,21\rangle$ having fixed $t(H)$ and $r(H)$.

$t(H)\backslash r(H)$	0	1	2	3	4	5	6	7	8	9	10	*Total*
0	1	0	0	0	0	0	0	0	0	0	0	1
1	0	4	1	2	2	1	0	0	0	0	0	10
2	0	0	6	3	6	7	2	3	0	0	0	27
3	0	0	0	4	3	2	8	7	6	1	0	31
4	0	0	0	0	1	0	2	3	3	3	1	13
Total	1	4	7	9	12	10	12	13	9	4	1	82

4 Weierstrass semigroups attained by ramification points on double covers of plane curves of degree 6

Theorem ([6]). *Let $d_2(H) = \langle 5,6\rangle$ or $\langle 5,9,13,17,21\rangle$. Assume $n = \min\{h \in H \mid h \text{ is odd}\} \geqq 21$. We set $r = r(H)$ and $t = t(H)$. Let*

$$H = 2d_2(H) + \langle n, n+2l_1, \dots, n+2l_t\rangle.$$

Suppose that there exists a pointed curve (C, P) with $H(P) = d_2(H)$ and r points $Q_1, \dots, Q_r$ of C, distinct from P such that
i) $h^0(Q_1 + \cdots + Q_r) = 1$,
ii) *$nP - 2(Q_1 + \cdots + Q_r)$ is linearly equivalent to a reduced divisor not containing P,*
iii) *$h^0(K - l_iP - Q_1 - \cdots - Q_r) = h^0(K - (l_i - 1)P - Q_1 - \cdots - Q_r)$ for any $i \in \{1, 2, \dots, t\}$*
where K is a canonical divisor on C.
Then H is attained by $H(\tilde{P})$ where $\tilde{P}$ is a ramification point on a double cover of C over P.

If n is sufficiently large, for example, larger than $2r + 20$, then the condition ii) is sastisfied. A canonical divisor K on the non-singular plain curve C of degree 6 is the intersection of a cubic curve and C. For points $P_1, \dots, P_s$ of C we see that $h^0(K - P_1 - \cdots - P_s)$ is the dimension of the cubics through the points $P_1, \dots, P_s$.

Let $H = 2\langle 5,6\rangle + \langle n, n+6\rangle$. Then $r(H) = 6$.

$$\begin{array}{cccccc}
(\rightarrow +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \odot & \times & \downarrow \\
(n) & \bullet & \times & \circ & \circ & +10 \\
 & (n+12) & \bullet & \circ & \circ & \\
\searrow +12 & & (n+24) & \bullet & \circ & \\
 & & & (n+36) & \bullet & \\
 & & & & (n+48) &
\end{array}$$

It suffices to show that $h^0(Q_1 + \cdots + Q_6) = 1$ and

$$h^0(K - 3P - Q_1 - \cdots - Q_6) = h^0(K - 2P - Q_1 - \cdots - Q_6).$$

We need to find a pointed curve (C, P) satisfying the above two conditions. Let C be the curve defined by the equation

$$z^3(yz^2 - y^3) + ax^3(yz^2 - x^3) + by^3(yz^2 + x^3 - 2y^3) = 0$$

where a and b are general elements of k and $P = (0 : 0 : 1)$. Let $Q_i = (\zeta^i : \zeta^{3i} : 1)$ for $i = 1, \ldots, 6$ where ζ is a primitive 6-th root of 1. We introduce the two cubics C_{31}: $yz^2 - x^3 = 0$ and C_{32}: $yz^2 + x^3 - 2y^3 = 0$, which are irreducible. Then we have $C_{31}.C_{32} = 3P+Q_1+\cdots+Q_6$, $C.C_{31} \geqq 3P+Q_1+\cdots+Q_6$ and $C.C_{32} \geqq 3P+Q_1+\cdots+Q_6$. The irreducibility of C_{31} and C_{32} means $h^0(Q_1 + \cdots + Q_6) = 1$. By Caylay-Bacharach Theorem which we show below we get

$$h^0(K - 2P - Q_1 - \cdots - Q_6) = h^0(K - 3P - Q_1 - \cdots - Q_6).$$

Theorem of Caylay-Bacharach. *Let X_1 and X_2 be two plane curves of degree d and e resp., meeting in a collection Γ of de points. Let C be a curve of degree $d+e-3$ containing all but one point of Γ. Then C contains all of Γ.*

Apply $d = e = 3$ to our case.

A numerical semigroup H is said to be *DCP* if there exists a double cover $\tilde{C}$ of a plane curve with a ramification point $\tilde{P}$ such that $H = H(\tilde{P})$.

Theorem 1 ([5]). *Let H be a numerical semigroup with $d_2(H) = \langle 5, 6\rangle$. Assume that the minimum odd integer n is larger than $2r(H) + 20$. Then H is DCP.*

But $H = 2\langle 5, 9, 13, 17, 21\rangle + \langle n, n + 16\rangle$ with $r(H) = 1$ is not DCP. Assume that H were DCP. Then $h^0(K - 8P - Q) = h^0(K - 7P - Q)$. But, there exists a cubic C_3 such that $C_3.C \geqq 7P + Q$ and $C_3.C \ngeqq 8P + Q$. This is a contradiction.

Theorem 2 ([5]). *Let H be a numerical semigroup with $d_2(H) = \langle 5, 9, 13, 17, 21\rangle$. Assume that the minimum odd integer n is larger than $2r(H) + 20$. Then we have the following table:*

DCP\\$r(H)$	0	1	2	3	4	5	6	7	8	9	10	*Total*
DCP	1	2	4	7	7	6	10	10	9	4	1	61
Not DCP	0	2	3	2	3	1	0	2	0	0	0	13
Unknown	0	0	0	0	2	3	2	1	0	0	0	8
Total	1	4	7	9	12	10	12	13	9	4	1	82

Remark. Recently we succeeded in determining whether the remaining eight unknown semigroups in the above table are DC or not ([5]).

References

[1] T. Harui and J. Komeda, *Numerical semigroups of genus eight and double coverings of curves of genus three*, Semigroup Forum. **89** (2014) 571–581.

[2] T. Harui and J. Komeda, *Numerical semigroups of genus seven and double coverings of curves of genus three*, Semigroup Forum. **90** (2015) 491–502.

[3] T. Harui and J. Komeda, *Numerical semigroups of genus six and double coverings of curves of genus three*, Semigroup Forum. **91** (2015) 601–610.

[4] S.J. Kim and J. Komeda, *Weierstrass semigroups on double covers of plane curves of degree 5*, Kodai Mathematical Journal. **38** (2015) 270–288.

[5] S.J. Kim and J. Komeda, *Weierstrass semigroups on double covers of plane curves of degree 6*, In preparation.

[6] J. Komeda, *On Weierstrass semigroups of double coverings of genus three curves*, Semigroup Forum. **83** (2011) 479–488.